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PHYSICAL PROPERTIES OF CRYSTALS

Isogyre Equation for Uniaxial and Biaxial Crystals

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Abstract—The isogyre equation determining the functional dependence between the coordinates of optical axes and isogyre points, valid for any cross sections in uniaxial and biaxial crystals, has been derived. It is used for plotting an isogyre and solving the inverse problem of determining the angle between the optical axes and the elements of orientation of a crystal's optical indicatrix.

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INTRODUCTION

The conoscopic method of studying crystals is based on the observation of the interference pattern obtained as a result of the transmission of a convergent light beam through a crystal plate located between crossed polarizer and analyzer. This method gives valuable information about some properties of a crystal, such as the number of optical axes, their dispersion, the angle between them, and the optical sign of the crystal.

Although great success has been achieved in recent decades in the simulation and interpretation of interference patterns, the level of applications of the conoscopic method (if we speak about a conoscope of a polarization microscope) corresponds to the beginning of the 20th century. The most significant gap is the absence of clear unambiguous understanding of the relationship between the orientation of crystal optical axes and the shape of dark fringes (isogyres) in the conoscopic pattern. This relationship can be described by an isogyre equation. Having this equation, one can determine the isogyre position in the field of view of the conoscope from the input parameters (the known directions of the crystal optical axes) and solve the inverse problem (which is more valuable for practical purposes), i.e., calculate the angle between the optical axes and the elements of orientation of the optical indicatrix from the coordinates of isogyre points.

After the first observations of interference patterns (last quarter of the 19th century), many researchers tried to explain their formation and plot isogyres via different techniques: calculations based on the hyperbola equation [1] with the use of auxiliary lines showing the direction of light oscillations in a crystal (skiodromes) [2], the simplified Fresnel rule [3], and solution of the problem of determining the directions of light oscillations in a stereographic projection [4].

In [5, 6], an isogyre is considered as a locus of an interference pattern, in which the directions of light oscillations are parallel to the main cross sections of Nicols. This concept of an isogyre contradicts an evident fact: in the case of incidence of oblique rays, the projections of two mutually perpendicular vectors of light oscillations in the crystal on the plane of a conoscopic pattern form in the general case skew angles, which cannot be aligned with the two mutually perpendicular main cross sections of Nicols.

In [7], an isogyre obtained by calculation of the intensity of passing light for a set of points of the conoscopic pattern was observed on a display. The main refractive indices N_g, N_m, and N_p and the elements of orientation of the plane cutting the optical indicatrix in the direction parallel to the plane of the crystal plate (three more parameters) were used as the initial data.

In [8, 9], the theory of formation of conoscopic patterns and the effect of different types of polarized radiation and a change in the angle between polarizer and analyzer on their shape were described and the criteria for the difference between the conoscopic patterns of optically active and inactive crystals were established. Unfortunately, not all these findings are applicable in the investigation of laps of rocks and minerals in a conoscope of a polarization microscope owing to the fuzziness of the conoscopic pattern and the small thickness of laps (as a result of which the phenomenon of rotation of the plane of polarization cannot be detected).

The purpose of this study is to derive the isogyre equation and apply it to some specific problems.

DERIVATION OF THE ISOGYRE EQUATION

Let us consider the formation of the isogyre of a conoscopic pattern. Let a crystal plate cut arbitrarily from a biaxial crystal be located between the crossed Nicols of a polarization microscope. In the rear focal plane of the objective, a system of bright and dark fringes forming the conoscopic pattern will be observed.

The light intensity at each point of the conoscopic pattern can be written as

$$B = I_0 \sin^2(\nu' - \nu'')\cos^2(\nu' + \nu''), \quad (1)$$

where I_0 is the incident light intensity and ν' and ν'' are the angles formed by the projections n' and n'' of the light oscillation vectors with the X axis (i.e., with the direction of light oscillations in one of the Nicols, see Fig. 1).

Formula (1) was derived in [10] for oblique incidence of light on the basis of the formula reported in [11] for normal incidence of light.

The projections n' and n'' form an angle α divided by the bisector MB oriented at an angle of 45° to the X axis. Hence, we have: $\nu' = 45° - \alpha/2$ and $\nu'' = 45° + \alpha/2$. The sum of the angles $\nu' + \nu'' = 90°$. Therefore, when the bisector makes an angle of 45° with the coordinate axes, the light intensity B is also zero. This is the condition for the isogyre passage through a specified point of the conoscopic pattern. On the basis of this condition, we can formulate the following definition: *an isogyre is a locus in the rear focal plane of the objective of a polarization microscope, in which the bisectors of the angles between the projections of the vectors of light oscillations in the crystal plate are oriented at an angle of* 45° *to the directions of light oscillations in Nicols* [10].

The angular coefficients k_1 and k_2 of the MN_1 and MN_2 lines have the form

$$k_1 = \tan\nu' = \tan(45° - \alpha/2),$$
$$k_2 = \tan\nu'' = \tan(45° + \alpha/2). \quad (2)$$

Since $\tan(45° - \alpha/2)\tan(45° + \alpha/2) = 1$ at any values of α, the equality

$$k_1 k_2 = 1 \quad (3)$$

is valid.

This laconic formula is a key to the derivation of the isogyre equation. It allows one to exclude from consideration physical characteristics, i.e., light intensity. Thus, we will deal below only with geometric quantities.

Formula (1) was used to reproduce a conoscopic pattern in the conoscope field of view [12]. The result obtained is similar to that reported in [7]; however, in contrast to [7], the orientations of the crystal optical axes were set as the initial data for calculation in [12]. The advantage of this approach is the simplicity of determining the directions of light oscillations in the crystal by the Fresnel rule.

The interference pattern observed in a conoscope is adequately reflected on the orthogonal projection of the spherical surface of the directions of light oscillations in a crystal. An arbitrary point M on a sphere of radius R is projected on the plane of projection of P to the point M' (Fig. 2a). The distance between the point M'

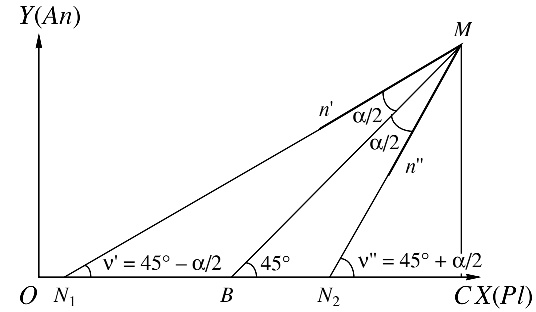

Fig. 1. Conditions for the passage of the isogyre through a specified point M of a conoscopic pattern. OX and OY are the Cartesian coordinate axes, with which the directions of light oscillations in the polarizer Pl and the analyzer An coincide.

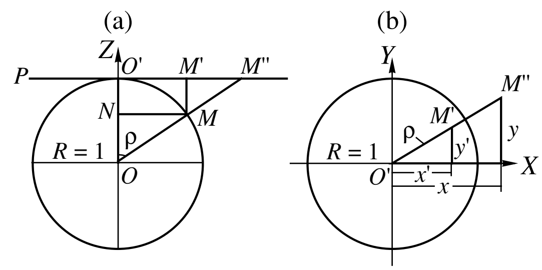

Fig. 2. Orthogonal and gnomonic projections: (a) the cross section perpendicular to the projection plane P and (b) the projection plane P lying in the drawing plane. X, Y, and Z are the Cartesian coordinate axes.

and the projection center O' is $O'M' = NM = R\sin\rho$, where ρ is the polar distance of the point M, which is equal to the length of the arc $O'M$. If we assume that $R = 1$, $O'M' = \sin\rho$. The field of the orthogonal projection is limited by the circle with the radius equal to the radius of the projected sphere.

In the gnomonic projection, the image of the point M is obtained at the intersection of the continuation of the radius OM with the plane of the projection P. The distance $O'M''$ between the point M'' and the center O' of the projection is $\tan\rho$. The projection field is not limited by anything and tends toward infinity at $\rho = 90°$.

It should be noted that the conoscopic pattern observed on a screen, obtained with a point light source, corresponds to the gnomonic projection.

Recalculation of the gnomonic coordinates x, y to the orthogonal coordinates x', y' (Fig. 2b) and vice versa is performed by the formulas

$$x' = \frac{x}{\sqrt{x^2 + y^2 + 1}}, \quad y' = \frac{y}{\sqrt{x^2 + y^2 + 1}},$$
$$x = \frac{x'}{\sqrt{1 - x'^2 - y'^2}}, \quad y = \frac{y'}{\sqrt{1 - x'^2 - y'^2}}. \quad (4)$$

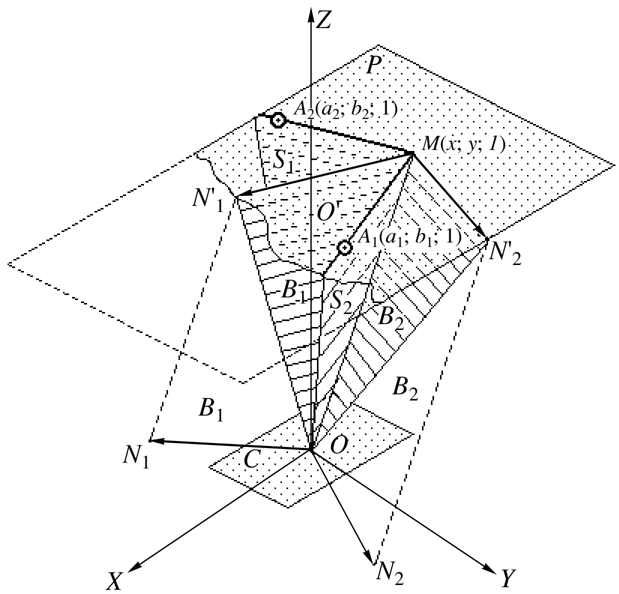

Fig. 3. Axonometric projection of the main optical cross sections and the planes of light oscillations in a biaxial crystal.

The oblique light ray OM, after passing through the crystal plate C, intersects the projection plane P at the point M with the gnomonic coordinates x, y, 1 (Fig. 3). The applicate z is equal to unity since it is arbitrarily assumed that the projection plane is spaced from the coordinate plane XOY by $R = 1$ (Fig. 2). Two mutually perpendicular light oscillation vectors $\mathbf{N}_1$ and $\mathbf{N}_2$ give projections N_1' and N_2' on the P plane (Fig. 3). It is necessary to determine the direction (angular coefficients) of these projections if the gnomonic coordinates of the points $A_1(a_1, b_1, 1)$ and $A_2(a_2, b_2, 1)$ of intersection of the optical axes with the sphere are known.

According to the Fresnel theorem, light oscillations in biaxial crystals are directed along the bisectors of the angles between the main cross sections of the indicatrix. The main cross section of an indicatrix is the plane passing through the normal to the wave and optical axis of the crystal. The indicatrix of a biaxial crystal obviously has two main cross sections, S_1 and S_2. Their equations can be written in the general form as

$$A_{1,2}x + B_{1,2}y + C_{1,2}z = 0, \tag{5}$$

where A, B, and C are the coefficients of equations and x, y, and z are the coordinates of points in the plane in 3D space.

The main cross sections S_1 and S_2 pass through the points with known gnomonic coordinates (in parentheses): the origin of coordinates $O(0, 0, 0)$, the point $M(x, y, 1)$, and the points $A_1(a_1, b_1, 1)$ and $A_2(a_2, b_2, 1)$. Coefficients of Eq. (5), which are calculated from the coordinates of these points, have the following values (it is taken into account for C_1 and C_2 that $z = 1$):

$$A_{1,2} = b_{1,2} - y, \quad B_{1,2} = x - a_{1,2}, \qquad C_{1,2} = -(A_{1,2}x + B_{1,2}y). \tag{6}$$

Light oscillations in a crystal occur in the planes B_1 and B_2 of the bisectors of the dihedral angles formed by the main cross sections S_1 and S_2 (the plane B_2 is perpendicular to the plane B_1). Both planes enter the bundle of planes; their equations can be written, respectively, as

$$(A_1 + \lambda A_2)x + (B_1 + \lambda B_2)y + (C_1 + \lambda C_2)z = 0, \qquad (A_1 - \lambda A_2)x + (B_1 - \lambda B_2)y + (C_1 - \lambda C_2)z = 0, \tag{7}$$

where λ is a coefficient.

After writing the expressions for cosines of the angles between B_1 and S_1 and S_2 and equating their right-hand sides to each other, we obtain

$$\lambda = ((A_1^2 + B_1^2 + C_1^2)/(A_2^2 + B_2^2 + C_2^2))^{1/2}. \tag{8}$$

Thus, we have Eqs. (7) of two mutually perpendicular planes B_1 and B_2, where light oscillations occur (vectors $\mathbf{ON}_1$ and $\mathbf{ON}_2$). Lines of intersection of these planes with the plane P form a gnomonic projection of the light oscillation vectors ON_1' and ON_2'. Substituting $z = 1$ into (7), we obtain equations of the straight lines ON_1' and ON_2':

$$y_1 = -[x(A_1 + \lambda A_2) + (C_1 + \lambda C_2)]/(B_1 + \lambda B_2), \qquad y_2 = -[x(A_1 - \lambda A_2) + (C_1 - \lambda C_2)]/(B_1 - \lambda B_2). \tag{9}$$

The angular coefficients k_1 and k_2 of these lines are, obviously,

$$k_1 = -(A_1 + \lambda A_2)/(B_1 + \lambda B_2), \qquad k_2 = -(A_1 - \lambda A_2)/(B_1 - \lambda B_2). \tag{10}$$

Using successively (10), (8), and (6), we find from (3) that

$$bx^3 + ay^3 - ax^2y - bxy^2 - cx^2 - cy^2 + 2(d-2)xy + 2bx + 2ay - 2c = 0, \tag{11}$$

where x, y are the gnomonic coordinates of isogyre points,

$$a = a_1 + a_2, \quad b = b_1 + b_2, \qquad c = a_1b_2 + a_2b_1, \quad d = a_1a_2 + b_1b_2. \tag{12}$$

Thus, we have derived a mathematical model of isogyre in the form of a third-power equation in gnomonic coordinates. It can be transformed into the cubic equation $y^3 + py^2 + qy + r = 0$, which has an analytic solution [13].

Replacing x and y with their values from formulas (4), we can transform Eq. (11) into the "actual" isogyre equation corresponding to the isogyre shape observed in practice. As a result, a cumbersome equation of sixth order with 16 terms is obtained. It has no analytic solution and cannot be used in practice. For this reason, it is necessary to calculate the coordinates of isogyre points along the specified directions of optical axes via gnomonic equation (11) and recount the obtained results into the orthogonal coordinates x' and y' by formula (4).

It should be noted that this recalculation is not required when a conoscopic pattern obtained with a point light source is projected on the screen. In this case, the isogyre equation in its initial form can be applied to the conoscopic pattern.

Results of calculation of the isogyre according to Eq. (11) in different cross sections of uniaxial and biaxial crystals for different angles of rotation of the microscope table showed that the isogyre displayed on a screen completely corresponds to the conoscopic pattern observed in practice.

The results of the construction of the axial line of the isogyre on the total orthogonal projection at a conoscopic angle of 90° (Fig. 4) are of particular interest. These data make it possible to reconstruct the virtual conoscopic pattern beyond the conoscope field of view, where the pattern cannot be observed using conventional objectives, because of their limited aperture. It has been revealed that the cross bars of a uniaxial crystal look straight-line only within the conoscope field of view with a $60^{\times}$ objective (small circle within a large circle, Fig. 4a). Beyond this field, two bars merge at the boundary of the projection at the point spaced by 45° from the coordinatea axes in the quadrant where the optical axis is located. The two other ends of the bars undergo bending and rest on the boundary of the projection in two opposite quadrants, at the points equidistant from the coordinate axes.

In the cross section of a biaxial crystal perpendicular to the plane of optical axes, at a certain orientation of the crystal, the isogyre has the shape of a double cross (Fig. 4b).

In a skew cross section of a biaxial crystal, the presence of the third branch of the isogyre is surprising. This branch has never been observed in practice, since it is located at the boundary of the projection outside the conoscope field of view (Fig. 4c). The largest nearby branch *1–2* passes through the point of emergence of the optical axis A_1, which is located closer to the projection center than the point A_2 corresponding to the second optical axis, with which the far branch *3–5* is related. The third branch *4–6* does not pass through the points A_1 and A_2. The ends of all three isogyre branches rest on the boundary circle. From six isogyre ends, four (*1*, *2*, *3*, and *4*) are rigidly bound to the fixed points spaced from the coordinates axes by 45°. These points are present in the conoscopic patterns of all cross sections of the crystals; they are immobile at the rotation of the microscope table. In contrast, ends *5* and *6* are mobile. Their position depends on the crystal cross section and the angle of rotation of the crystal plate. It can be seen in the figure that the isogyre, resting on the projection boundary, is not interrupted but continues at the opposite point on the boundary circle.

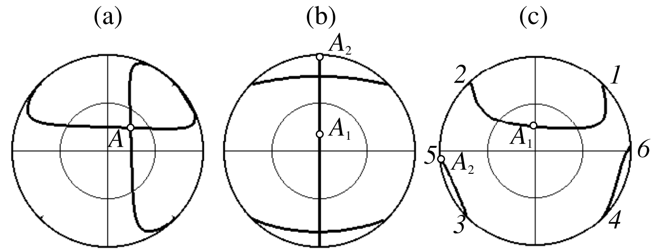

Fig. 4. The axial isogyre lines on the complete orthogonal projection (computer reproduction): (a) the skew cross section of a uniaxial crystal, (b) the double-cross figure for a biaxial crystal in the cross section perpendicular to the plane of optical axes, and (c) the isogyre with three branches in the skew cross section of a biaxial crystal with $2V = 85°$.

Figure 5 shows the behavior of the isogyre of the skew cross section of a biaxial crystal on the complete orthogonal projection during crystal rotation. In the initial position (the crystal is extinguished), the nearby branch of the isogyre passes through the projection center ($\omega = 0°$). Two other branches are located at different sides from it.

Under crystal rotation by an angle in the range $0° \leq \omega \leq 90°$, the nearby branch of the isogyre consecutively forms a skew cross, while connecting first with the branch having no axis ($\omega = 4.7°$) and then with the far branch ($\omega = 69°$). Thus, as a result of the crystal rotation by 360°, the cross is formed eight times. When conventional objectives are used, the cross figure arises only four times during the complete rotation of the table.

The nearby branch of the isogyre, in any position, always rests on the fixed points and only this branch can cross the projection center. The two other branches rest by one end on a fixed point and by the other end on a mobile point, which changes its location during crystal rotation, while moving along the circle limiting the projection field.

EXAMPLE OF THE USE OF THE ISOGYRE EQUATION

As an example of application of the isogyre equation in practice, let us describe the technique of conoscopic measurement of the angle of inclination of the optical axis of a uniaxial crystal, as proposed in [14] and used for routine measurements of the orientation of the optical axes in quartz grains in order to detect the preferential orientation of this mineral in laps of rocks (microstructural analysis) [15]. In the initial position (the crystal is extinguished), the optical axis A is on the coordinate axis Y outside the conoscope field of view; therefore, it is inaccessible for direct observation and measurement of its inclination ρ_A (the arc OA) using an

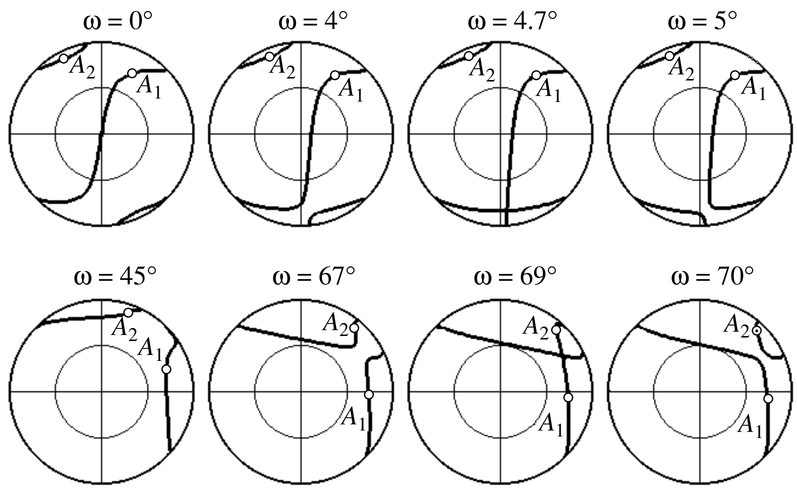

Fig. 5. Isogyre on the complete orthogonal projection and its behavior under rotation of the microscope table for the skew cross section of a biaxial crystal with $2V = 47°$. ω is the angle of rotation of the crystal plate from the position of the crystal extinguishment.

ocular micrometer (Fig. 6a). The gnomonic coordinates of the optical axis in this position are $a_1 = 0$ and $b_1 = \tan\rho_A$.

As a result of the rotation of the microscope table by some angle ω, the isogyre shifts to the right and reaches the mark M on the X axis (Fig. 6b). One of the divisions of the micrometer ocular scale can be used as a mark. Its gnomonic coordinates are $x = \tan\rho_M$, $y = 0$, where ρ_M is the angular distance between the mark and the center O (the polar distance), which is known from the calibration of the ocular micrometer scale. The coordinates of the point of emergence of the optical axis in this position are

$$a_1 = \tan\rho_A \sin\omega, \quad b_1 = \tan\rho_A \cos\omega. \tag{13}$$

Since the isogyre point M lies on the X axis ($y = 0$), Eq. (11) takes the form $(x^2 + 2)(bx - c) = 0$. In this case, Eq. (11) has a single real root

$$x = c/b. \tag{14}$$

For a uniaxial crystal, it follows from (12) that $b = 2b_1$ and $c = 2a_1b_1$; hence, we find from (14) that $x_M = a_1$. It can be seen from this equality that, under rotation of the microscope table, the abscissas of the optical axis A of the uniaxial crystal and the point of intersection of the isogyre with the X axis are always equal to each other; therefore, we find from (13) that

$$\tan\rho_A = \tan\rho_M / \sin\omega. \tag{15}$$

We have also solved a similar, but much more difficult, problem of determining the orientation of the optical axes for skew cross sections of a biaxial crystal. To solve this problem, it is necessary to measure the coordinates x and y of three points of the isogyre; substitute them into Eq. (11); solve the system of three equations; and find the parameters a, b, c, and d. Knowing these parameters and using (12), one can calculate the gnomonic coordinates of the optical axes. These coordinates are recalculated into spherical coordinates and, furthermore, used to determine graphically (on a stereographic grid) or analytically (by formulas of spherical trigonometry) the angle of the optical axes and the orientation of the optical indicatrix.

It is known that the application of the conoscopic method is related to a number of limitations restricting its potential. Specifically:

(i) Only the cross sections of the crystal oriented perpendicular to the sharp bisector can be used to measure the angle $2V$ between the optical axes.

(ii) The measured angle $2V$ should not exceed 50°–60°.

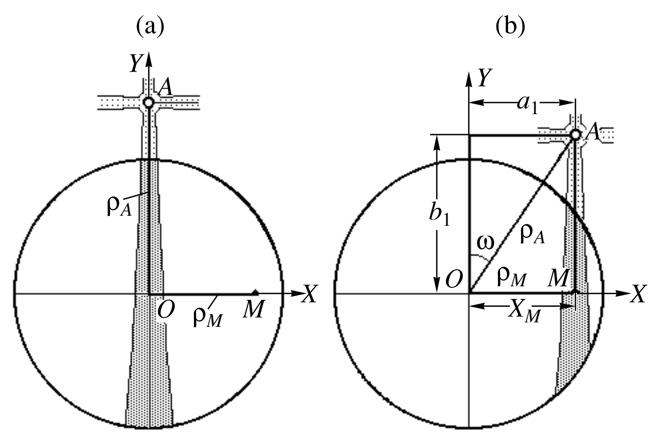

Fig. 6. Measurement of the inclination of the optical axis of a uniaxial crystal (gnomonic projection): (a) initial position (the crystal extinguished) and (b) position after the rotation of the microscope table by an angle ω.

(iii) The orientation of the optical indicatrix cannot be determined.

The use of the isogyre equation makes it possible to remove these limitations, as a result of which the conoscopic method allows obtainment of results comparable with those obtained on a Fedorov universal table.

CONCLUSIONS

As a result of the investigations performed, a gnomonic isogyre equation has been derived. In contrast to the generally accepted concept, this equation turned out to correspond not to a hyperbola but to a more complex line of the third order. The axial line of the isogyre has been plotted in the maximum angle range (180°). Unusual shapes of the isogyre and specific features of its behavior during crystal rotation have been revealed, which previously were not described in the literature and were not observed in operation with conventional objectives owing to their limited aperture.

It is proposed to use the isogyre equation as a basis for development of new techniques for measuring the angle of optical axes and the elements of orientation of the optical indicatrix. Such techniques would significantly expand the potential of the conoscopic method.

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